Generic identifiability of tensor rank decompositions

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December 2, 2015

Joint work with L. Chiantini and G. Ottaviani.

Overview

1. Tensor decompositions
2. Generic identifiability
   - Secant varieties
   - Tangential weak defectivity
   - The Veronese case
   - The Segre case
3. An application
4. Conclusions

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Complexity of matrix multiplication

Consider the problem of multiplying 2 matrices of size 2×2:

$$M_2 : \mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$$

$$\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) \mapsto \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

This multilinear operator can be represented as a tensor

$$M_2 \in \mathbb{C}^4 \otimes (\mathbb{C}^4 \otimes \mathbb{C}^4)^* \cong \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4.$$

Complexity of matrix multiplication

The tensor $M_2 \in \mathbb{C}^{4\times 4\times 4}$ is given by

$$
\underset{c_{11}}{\begin{array}{cc} & \begin{array}{cccc} b_{11} & b_{12} & b_{21} & b_{22} \end{array} \\ \begin{array}{c} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{array} & \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & \\ & & & \end{bmatrix} \end{array}},
\underset{c_{12}}{\begin{array}{c} \begin{array}{cccc} b_{11} & b_{12} & b_{21} & b_{22} \end{array} \\ \begin{bmatrix} & 1 & & \\ & & & 1 \\ & & & \\ & & & \end{bmatrix} \end{array}},
$$

$$
\underset{c_{21}}{\begin{array}{cc} & \begin{array}{cccc} b_{11} & b_{12} & b_{21} & b_{22} \end{array} \\ \begin{array}{c} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{array} & \begin{bmatrix} & & & \\ & & & \\ 1 & & & \\ & & 1 & \end{bmatrix} \end{array}},
\underset{c_{22}}{\begin{array}{c} \begin{array}{cccc} b_{11} & b_{12} & b_{21} & b_{22} \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & 1 & & \\ & & & 1 \end{bmatrix} \end{array}}.
$$

Complexity of matrix multiplication

Letting

$$M_2 = \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$$

and

$$\mathbf{a} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix},$$

then $C = AB$ is equivalent to

$$\mathbf{c} = (I \otimes \mathbf{a} \otimes \mathbf{b})^T M_2 = \sum_{i=1}^{r} (\mathbf{y}_i^T \mathbf{a})(\mathbf{z}_i^T \mathbf{b})\mathbf{x}_i.$$

The bilinear complexity is defined as r.

Complexity of matrix multiplication

The decomposition

$$M_2 = \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$$

is called a tensor rank decomposition if r is minimal.

As M_2 contains only 8 nonzero elements, it follows that $r \leq 8$.

In a landmark paper, Strassen [S1969] proved that M_2 has bilinear complexity $r = 7$. This leads to an $n^{\log_2 7} \approx n^{2.8}$ algorithm for $n \times n$ matrix multiplication.

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Open problem: What is the rank of M_3? $(15 \leq \mathrm{rank}(M_3) \leq 21)$

Complexity of polynomial evaluation

Waring's problem for polynomials asks to write a complex homogeneous polynomial as

$$\begin{aligned} f(x_0, x_1, \ldots, x_n) &= \sum_{0 \le i_1 \le i_2 \le \cdots \le i_d \le n} f_{i_1,\ldots,i_d} x_{i_1} x_{i_2} \cdots x_{i_d} \\ &= \sum_{i=1}^{r} (a_{i,0} x_0 + a_{i,1} x_1 + \cdots + a_{i,n} x_n)^d, \end{aligned}$$

where $r \in \mathbb{N}$ is minimal.

Complexity of polynomial evaluation

The multiplicative complexity of evaluating f when it is given by coefficients $f_{i_1,\ldots,i_d}$ is

$$d\binom{n+d}{d} \text{ multiplications,}$$

whereas a polynomial given by its Waring decomposition requires at most

$$r(n+d) \text{ multiplications.}$$

Waring's decomposition

Let P contain all permutations of $\{1, 2, \ldots, r\}$.

The space of complex polynomials of homogeneous degree d in $n+1$ variables $\mathbb{P}S^d\mathbb{C}^{n+1}$ is isomorphic to the projectivization of

$$\{a_{i_1,i_2,\ldots,i_d} = a_{i_{\sigma_1},i_{\sigma_2},\ldots,i_{\sigma_d}} \mid \sigma \in P\} \subset \mathbb{C}^{n+1\times\cdots\times n+1},$$

consisting of symmetric tensors in $\mathbb{C}^{n+1} \otimes \cdots \otimes \mathbb{C}^{n+1}$.

Waring's decomposition

Considered as a tensor decomposition problem, Waring's problem is equivalent to finding

$$\mathfrak{F} = \sum_{i=1}^{r} \mathbf{a}_i \otimes \cdots \otimes \mathbf{a}_i = \sum_{i=1}^{r} \mathbf{a}_i^{\otimes d},$$

where $\mathfrak{F} \in L$ is the symmetric tensor corresponding to f, and where $\mathbf{a}_i \in \mathbb{C}^{n+1}$.

If $r \in \mathbb{N}$ is minimal, then the above expression is called a Waring's decomposition.

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Segre and Veronese

Definition

The projectivization of the set of rank-1 tensors in $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$ is a projective variety called the Segre variety, denoted by

$$\mathcal{S} = \operatorname{Seg}(\mathbb{P}\mathbb{C}^{n_1} \times \cdots \times \mathbb{P}\mathbb{C}^{n_d}).$$

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$$\mathcal{V} = \mathbb{P}v_d(\mathbb{C}^{n+1}).$$

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Secant varieties

Consider a "sufficiently nice" projective variety $\mathcal{X} \subset \mathbb{P}^N$, i.e.,

- irreducible,
- not contained in a hyperplane,
- over $\mathbb{C}$.

The (irreducible) *r*-secant variety of $\mathcal{X}$ is defined as

$$\sigma_r(\mathcal{X}) = \overline{\bigcup_{p_1,\ldots,p_r \in \mathcal{X}} \langle p_1, \ldots, p_r \rangle},$$

where the overline denotes the Zariski-closure.

Secant varieties

Tensor rank decompositions of length r, i.e.,

$$\sum_{i=1}^{r} \mathbf{a}_i^1 \otimes \mathbf{a}_i^2 \otimes \cdots \otimes \mathbf{a}_i^d$$

constitute a dense constructible subset of $\sigma_r(\mathcal{S})$.

Waring decompositions of length r, i.e.,

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constitute a dense constructible subset of $\sigma_r(\mathcal{V})$.

Secant varieties

Although the secant varieties of $\mathcal{X}$ are an elementary construction, we known very little about

- dimension,
- secant order, and
- equations (set, ideal, scheme),

even for classic families of varieties such as

- Veronese,
- Segre,
- Segre–Veronese,
- Chow,
- ...

Main question

What is the r-secant order of $\mathcal{X}$?

How many rank-r decompositions does a tensor admit?

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$\uparrow$

What is the dimension of $\sigma_r(\mathcal{X})$?

Dimensions of secant varieties

The dimension of the r-secant variety of $\mathcal{X} \subset \mathbb{P}^N$ satisfies

$$\dim \sigma_r(\mathcal{X}) \leq \min\{r(\dim \mathcal{X} + 1) - 1, N\}.$$

One expects that equality holds. If it does not, $\sigma_r(\mathcal{X})$ is called defective.

So far, the completely understood families of classic varieties are:

- the Veronese variety [AH1995], and
- the tangential of the Veronese variety [AV2015].

Dimensions of secant varieties and generic rank

Note that we have an ascending chain of secant varieties:

$$\mathcal{X} = \sigma_1(\mathcal{X}) \subsetneq \sigma_2(\mathcal{X}) \subsetneq \cdots \subsetneq \sigma_{\bar{r}-1}(\mathcal{X}) \subsetneq \sigma_{\bar{r}}(\mathcal{X}) = \sigma_{\bar{r}+1}(\mathcal{X}) = \mathbb{P}^N.$$

Definition

The value $\bar{r}$ is called the generic $\mathcal{X}$-rank, a value $r < \bar{r}$ is called subgeneric, and $r > \bar{r}$ is called supergeneric.

Basic results on number of decompositions

Lemma (Defective case)

The generic tensor $p \in \sigma_r(\mathcal{X})$ on a defective secant variety admits ∞-many decompositions.

Lemma (Supergeneric case)

Let r be supergeneric. Then, the generic tensor $p \in \sigma_r(\mathcal{X})$ admits ∞-many decompositions.

Lemma (Subgeneric case)

Let r be subgeneric, and let $\sigma_r(\mathcal{X})$ be nondefective. Then, the generic tensor $p \in \sigma_r(\mathcal{X})$ admits finitely many decompositions.

The Veronese variety is mostly nondefective

Theorem (Alexander and Hirschowitz, 1995)

The only r-defective secant varieties of the dth Veronese embedding of $\mathbb{C}^{n+1}$ are:

- *symmetric matrices, i.e., $d = 2$,*
- $(d, n, r) = (3, 4, 7)$,
- $(d, n, r) = (4, 2, 5)$,
- $(d, n, r) = (4, 3, 9)$, *and*
- $(d, n, r) = (4, 4, 14)$.

Hence, the generic symmetric tensor of subgeneric rank has finitely many Waring decompositions.

The Segre variety is expected to be mostly nondefective

Conjecture (Abo, Ottaviani, and Peterson, 2009)

The only defective Segre varieties are:

- *matrices, i.e.,* $d = 2$,
- $\mathcal{S}$ *is unbalanced:* $n_1 > 1 + \prod_{i=2}^{d} n_i - \sum_{i=2}^{d}(n_i - 1)$,
- $\mathcal{S} = \mathrm{Seg}(\mathbb{PC}^n \times \mathbb{PC}^n \times \mathbb{PC}^2 \times \mathbb{PC}^2)$,
- $\mathcal{S} = \mathrm{Seg}(\mathbb{PC}^n \times \mathbb{PC}^n \times \mathbb{PC}^3)$, *$n$ odd,*
- $\mathcal{S} = \mathrm{Seg}(\mathbb{PC}^4 \times \mathbb{PC}^4 \times \mathbb{PC}^3)$, *and*

Hence, the generic tensor of subgeneric rank is expected to have finitely many tensor rank decompositions.

Main question

What is the r-secant order of $\mathcal{X}$?
A natural number!

How many rank-r decompositions does a tensor admit?
Finitely many!

$\uparrow$

What is the dimension of $\sigma_r(\mathcal{X})$?
As expected!

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Secant order

Definition (Chiantini and Ciliberto, 2006)

The r-secant order of $p \in \mathcal{X} \subset \mathbb{P}^N$ is the number of distinct linear spaces

$$\langle p_1, p_2, \ldots, p_r \rangle, \quad p_i \in \mathcal{X}$$

that contain p.

If the generic tensor $p \in \sigma_r(\mathcal{X})$ has r-secant order equal to one, then $\mathcal{X}$ is called generically r-identifiable.

Tangent spaces and Terracini's lemma

Definition (Tangent space)

For a smooth manifold $\mathcal{X}$, the tangent space to $\mathcal{X}$ at $p = p(0)$ is defined geometrically as

$$\mathrm{T}_p\mathcal{X} = \{\tfrac{d}{dt}p(t)\big|_0 \mid p(t) \in \mathcal{X}\}.$$

Lemma (Terracini, 1915)

Let $p \in \langle p_1, p_2, \ldots, p_r\rangle$ with $p_i \in \mathcal{X}$ be generic. Then,

$$\mathrm{T}_p\sigma_r(\mathcal{X}) = \langle \mathrm{T}_{p_1}\mathcal{X}, \mathrm{T}_{p_2}\mathcal{X}, \ldots, \mathrm{T}_{p_r}\mathcal{X}\rangle.$$

A criterion for generic identifiability

Lemma (Chiantini, Ottaviani, and V, 2014)

Let $p \in \langle p_1, p_2, \ldots, p_r \rangle$ with $p_i \in \mathcal{X}$ be generic. If $\mathcal{X}$ is not generically r-identifiable, then the r-contact locus

$$\mathcal{C}_r = \{q \mid \mathrm{T}_q\mathcal{X} \subset \mathrm{T}_p\sigma_r(\mathcal{X}) = \langle \mathrm{T}_{p_1}\mathcal{X}, \mathrm{T}_{p_2}\mathcal{X}, \ldots, \mathrm{T}_{p_r}\mathcal{X} \rangle\}$$

contains a curve passing through each p_i.

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contains a curve passing through each p_i.

Sketch of proof.

Let $p \in \langle p_1, p_2, \ldots, p_r \rangle \cap \langle q_1, q_2, \ldots, q_r \rangle$ with r minimal. Then

$$\langle \mathrm{T}_{p_1}\mathcal{X}, \ldots, \mathrm{T}_{p_r}\mathcal{X} \rangle = \langle \mathrm{T}_{q_1}\mathcal{X}, \ldots, \mathrm{T}_{q_r}\mathcal{X} \rangle,$$

for otherwise p is singular. By nontriviality and genericity, all $q_i \notin \{p_1, \ldots, p_r\}$. Let p vary in $\langle q_1, \ldots, q_r \rangle$, and then all $p_i \in \mathcal{X}$ move with $\mathrm{T}_{p_i}\mathcal{X} \subset \mathrm{T}_p\sigma_r(\mathcal{X})$. $\square$

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Theorem (Chiantini, Ottaviani, and V, 2015)

Let $\mathcal{V}$ be the dth Veronese embedding of $\mathbb{C}^{n+1}$ in $\mathbb{P}S^d\mathbb{C}^{n+1}$. Then, $\mathcal{V}$ is generically r-identifiable for all subgeneric r, unless

- $d = 2$,
- $(d, n, r) = (6, 2, 9)$,
- $(d, n, r) = (4, 3, 8)$, *or*
- $(d, n, r) = (3, 5, 9)$.

In the exceptional cases, there are generically 2 Waring decompositions.

All exceptional cases arise because of the existence of a (unique) elliptic normal curve passing through r generic points of $\mathbb{P}^n$.

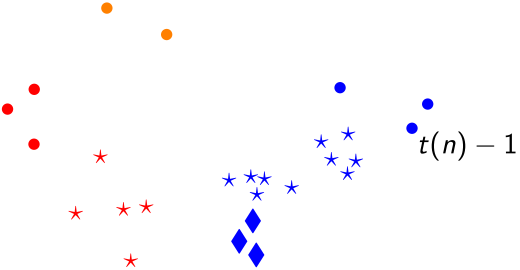
Proof idea of Brambilla and Ottaviani (2008)

For $n \not\equiv 2 \mod 3$ the generic rank is

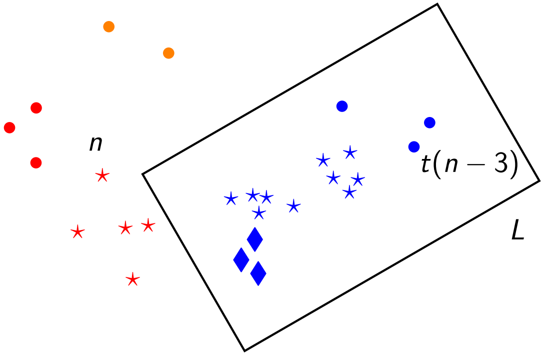
$$t(n) = \frac{(n+3)(n+2)}{6}.$$

This suggests specializing points on codimension 3 subspaces.

Proof idea of Brambilla and Ottaviani (2008) — Step 0



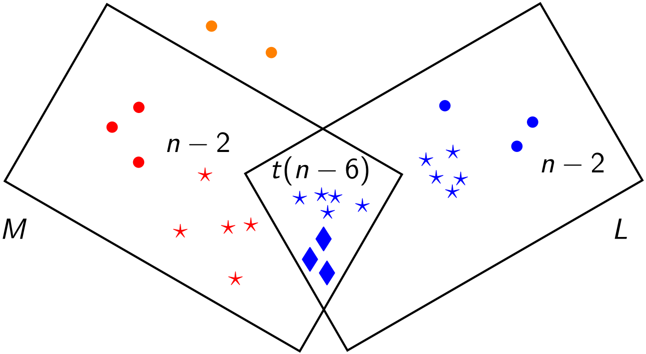
Proof idea of Brambilla and Ottaviani (2008) — Step 1



$$0 \to I_{L\cup \mathbf{x},\mathbb{P}^n}(3) \to S_{\mathbb{P}^n}(3) \to S_L(3)$$

L is a codimension 3 generic subspace of $\mathbb{PC}^{n+1}$.

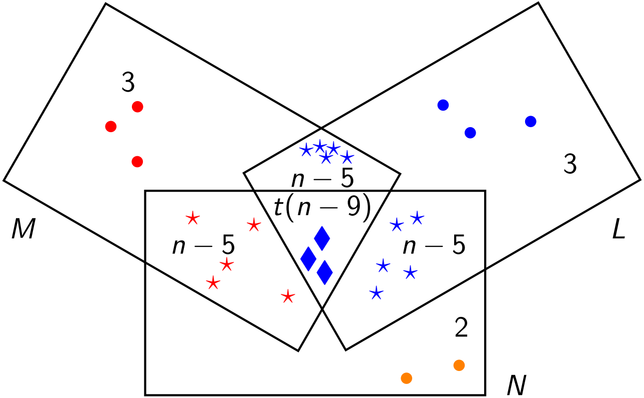
Proof idea of Brambilla and Ottaviani (2008) — Step 2



$$0 \to I_{L \cup M \cup \mathbf{x}, \mathbb{P}^n}(3) \to I_{L \cup \mathbf{x}, \mathbb{P}^n}(3) \to I_{(L \cup \mathbf{x}) \cap M, M}(3)$$

L and M are codimension 3 generic subspaces of $\mathbb{PC}^{n+1}$.

Proof idea of Brambilla and Ottaviani (2008) — Step 3



$$0 \to I_{L\cup M\cup N,\mathbb{P}^n}(3) \to I_{L\cup M,\mathbb{P}^n}(3) \to I_{(L\cup M)\cap N,N}(3)$$

L, M, and N are codimension 3 generic subspaces of $\mathbb{PC}^{n+1}$.

Proof idea of Brambilla and Ottaviani (2008)

The base cases of this inductive proof are proved by computer in Macaulay2.

The case $n \equiv 2 \mod 3$ is more difficult and requires induction on subspaces of codimension 3, 4, and 4, as well as on subspaces of codimension 4, 4, and 3.

Main question

What is the r-secant order of $v_d(\mathbb{C}^{n+1})$?

1!

How many Waring decompositions does a rank-r symmetric tensor admit?

1!

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An algorithm proving generic identifiability

As we do not know the dimension of the secant varieties of Segre varieties, we do not aim at formal proofs in this case.

Consider the tangentially r-contact variety:

$$\mathcal{C}_r = \{p \in \mathcal{S} \mid \mathrm{T}_p\mathcal{S} \subset T = \langle \mathrm{T}_{p_1}\mathcal{S}, \mathrm{T}_{p_2}\mathcal{S}, \ldots, \mathrm{T}_{p_r}\mathcal{S}\rangle\}$$

for some generic fixed $p_i \in \mathcal{S}$.

By the main Lemma of [COV2014],

$\mathcal{C}_r$ is zero-dimensional $\rightarrow$ the Segre variety $\mathcal{S}$ is r-identifiable.

An algorithm proving generic identifiability: Step 1

Sample r random points on the Segre variety. That is, choose

$$p_i = \mathbf{a}_i^1 \otimes \mathbf{a}_i^2 \otimes \cdots \otimes \mathbf{a}_i^d,$$

and let T_i be a matrix representing $\mathrm{T}_{p_i}\mathcal{S}$. Then,

$$T = \begin{bmatrix} T_1 & T_2 & \cdots & T_r \end{bmatrix}.$$

With probability 1, $\mathrm{T}_p\sigma_r(\mathcal{S}) = \mathrm{Im}(T)$.

An algorithm proving generic identifiability: Step 2

Let

$$\Pi := \prod_{k=1}^{d} n_k, \quad \Sigma := \sum_{k=1}^{d} (n_k - 1), \text{ and } \quad \ell := \Pi - r\Sigma,$$

Note: ℓ is the (expected) codimension of $\sigma_r(\mathcal{S}) \subset \mathbb{P}^{\Pi-1}$.

Let $m(\cdot)$ be such that

$$x_{m(i_1,i_2,\ldots,i_d)} = a^1_{i_1} a^2_{i_2} \cdots a^d_{i_d} \quad \text{if} \quad \mathbf{x} = \mathbf{a}^1 \otimes \mathbf{a}^2 \otimes \cdots \otimes \mathbf{a}^d.$$

An algorithm proving generic identifiability: Step 2

Let $K = [\mathbf{k}_i]$ be a matrix representation of $(\operatorname{Im} T)^{\perp} \subset \mathbb{C}^{\Pi}$.

If $\operatorname{rank}(K) \neq \ell$, then $\mathcal{S}$ is probably r-defective.

Otherwise,

$$\begin{aligned} q_l(x_1, x_2, \ldots, x_\Pi) &= \sum_{i=1}^{\Pi} k_{i,l} x_i \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} k_{m(i_1,i_2,\ldots,i_d),l} x_{m(i_1,i_2,\ldots,i_d)} = 0 \end{aligned}$$

are the ℓ equations of T.

An algorithm proving generic identifiability: Step 3

Plugging in the parameterization of rank-1 tensors, we can write

$$q_l(\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^d) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} k_{m(i_1,i_2,\ldots,i_d),l} a^1_{i_1} a^2_{i_2} \cdots a^d_{i_d} = 0.$$

This gives equations for $T \cap \mathcal{S}$.

Deriving with respect to the parameterization of the Segre variety yields

$$\begin{bmatrix} \frac{\partial}{\partial \mathbf{a}} q_1(\mathbf{a}) \\ \frac{\partial}{\partial \mathbf{a}} q_2(\mathbf{a}) \\ \vdots \\ \frac{\partial}{\partial \mathbf{a}} q_\ell(\mathbf{a}) \end{bmatrix} = 0,$$

which are equations of $\mathcal{C}_r = T \cap \mathrm{T}\mathcal{S}$.

An algorithm proving generic identifiability: Step 4

To compute the dimension of $\mathcal{C}_r$, we construct the tangent space:

$$H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_\ell \end{bmatrix}$$

where

$$H_l = \frac{\partial^2}{\partial \mathbf{a} \partial \mathbf{a}} q_l(\mathbf{a}).$$

rank$(H) = \Sigma \rightarrow$ The Segre variety $\mathcal{S}$ is r-identifiable.

An identifiability conjecture

Conjecture: The generic $p \in \sigma_r(\mathrm{Seg}(\mathbb{PC}^{n_1} \times \cdots \times \mathbb{PC}^{n_d}))$ is r-identifiable if $r < \overline{r}$, unless

$(n_1, \ldots, n_d)$		type	reference
(m, n)	$r \geq 2$	defective	
$(4, 4, 3)$	$r = \overline{r} - 1$	defective	[AOP2009]
$(4, 4, 4)$	$r = \overline{r} - 1$	sporadic	[CO2012]
$(6, 6, 3)$	$r = \overline{r} - 1$	sporadic	[CMO2014]
$(n, n, 2, 2)$	$r = \overline{r} - 1$	defective	[AOP2009]
$(2, 2, 2, 2, 2)$	$r = \overline{r} - 1$	sporadic	[BC2013]
$n_1 \geq \eta$	$r \geq \eta$	defective	[BCO2013]

with $\eta = \prod_{k=2}^{d} n_k - \sum_{k=2}^{d}(n_k - 1)$.

A uniqueness conjecture

Using the linear algebra algorithm, the conjecture was verified in spaces of dimension up to $\Pi = 17500$.

[BCO2013] verified the conjecture in spaces up to dimension $\Pi = 100$ using Macaulay2.

The algorithm found a class of tangentially weakly defective varieties that are nevertheless identifiable.

Main question

What is the r-secant order of $\mathcal{S}$?

1!

How many rank-r decompositions does a tensor admit?

1!

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Blind source separation problem

Consider the relation

$$Y = MX$$

where

- $Y \in \mathbb{R}^{p \times N}$ are known **observed signals**,
- $M \in \mathbb{R}^{p \times p}$ is an unknown **mixing matrix**, and
- $X \in \mathbb{R}^{p \times N}$ are unknown **source signals**.

Can we recover X?

Blind source separation problem

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- $X \in \mathbb{R}^{p \times N}$ are unknown **source signals**.

Can we recover X?

No! There is an entire manifold of possible solutions:

$$Y = MX = (MA)(A^{-1}X), \quad A \text{ nonsingular.}$$

Blind source separation

Let's try again. Given $Y \in \mathbb{R}^{p\times N}$ in

$$Y = MX,$$

find $M \in \mathbb{R}^{p\times p}$ and $X \in \mathbb{R}^{p\times N}$.

Now assume the following:

- With the rows of X we associate a random vector $\mathbf{x}$.
- With the rows of Y we associate a random vector $\mathbf{y} = M\mathbf{x}$.
- Every column of X represents a sample of $\mathbf{x}$.
- Every column of Y represents the corresponding sample of $\mathbf{y}$.
- The random variates in $\mathbf{x}$ are (nearly) statistically independent.

Cumulants

Recall the definition of the cumulants of random variates with mean zero:

$$\begin{aligned} C(w) &= \mathrm{E}\{w\} \\ C(w,x) &= \mathrm{E}\{wx\} \\ C(w,x,y) &= \mathrm{E}\{wxy\} \\ C(w,x,y,z) &= \mathrm{E}\{wxyz\} - \mathrm{E}\{wx\}\mathrm{E}\{yz\} - \mathrm{E}\{wy\}\mathrm{E}\{xz\} \\ &\quad - \mathrm{E}\{wz\}\mathrm{E}\{xy\} \end{aligned}$$

For a random vector $x = [x_1\ x_2\ \cdots\ x_n]$, the second-order cumulant is the covariance matrix:

$$[C(x_i x_j)]_{i,j=1}^{n}$$

Cumulants are symmetric tensors

Consider the random vector $x = [x_1 \; x_2 \; \cdots \; x_p]$, and let

$$\mathfrak{C}_x^4 = [C(x_i x_j x_k x_l)]_{ijkl=1}^{p} \in \mathbb{R}^{p \times p \times p \times p}$$

be the fourth-order cumulant tensor. It is a symmetric tensor.

Blind source separation is symmetric tensor decomposition

The random variates in $\mathbf{x}$ are (nearly) statistically independent.

This is equivalent with a diagonal dth order cumulant:

$$\mathfrak{C}_x^d = \sum_{i=1}^{p} c_i \mathbf{e}_i^{\otimes d}.$$

If $\mathbf{y} = M\mathbf{x}$, then it is known that

$$\mathfrak{C}_y^4 = \sum_{i=1}^{p} c_i \cdot (M\mathbf{e}_i)^{\otimes 4}.$$

Thus, Waring's decomposition, if it is unique, recovers M.

The cocktail party problem

Suppose we have

- p persons in the room, speaking,
- p microphones spread over the room,
- we record sound at N discrete time steps.

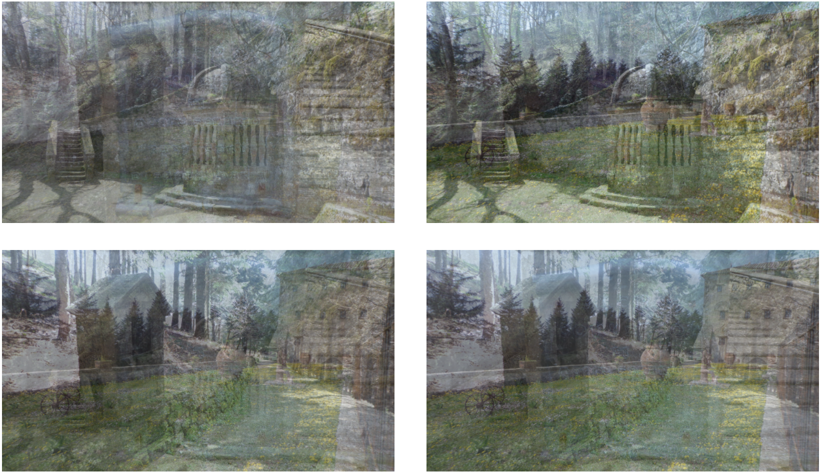
Because of the superposition of sound waves, we can model this as

$$Y = MX,$$

where,

- row i of X contains the sound produced by person i at each of the time steps.
- row i of Y contains the sound measured at microphone i at each of the time steps.

The cyclically (and seriously) broken camera problem



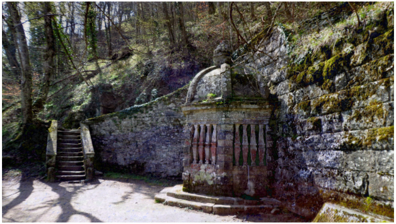
The cyclically (and seriously) broken camera problem

Recovered

Original

The cyclically (and seriously) broken camera problem

Recovered



Original

Overview

1. Tensor decompositions

2. Generic identifiability
   - Secant varieties
   - Tangential weak defectivity
   - The Veronese case
   - The Segre case

3. An application

4. Conclusions

Summary

We saw that

- tensor rank is related to algebraic complexity theory,
- symmetric tensors are generically identifiable,
- tensors are expected to be generically identifiable,
- and ...

Conclusions

… if your camera starts doing this:

Consider tensor decompositions!

Thank you for your attention!

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Further reading

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December 17 at **12h45** there will be a follow-up seminar at the Department of Computer Science in Auditorium **200A 00.225** entitled

Towards a perturbation theory for the tensor rank decomposition.